

Planar Semi-Martingales*

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Communicated by S. Watanabe

The concept of a semi-martingale is extended to processes with index set in the plane. The definitions of planar semi-martingales are similar to those of two parameter bounded variation. Necessary and sufficient conditions for a Doob-Meyer decomposition are obtained, and a maximal inequality and almost everywhere convergence theorem is given for planar semi-martingales in LlogL.

A process $\{X(t), \mathcal{F}(t), 0 \leq t \leq 1\}$ is a semi-martingale if

$$\sup \left\{ E \left(\sum_{i=0}^{n-1} |E(X(t_{i+1}) - X(t_i) | \mathcal{F}(t_i))| \right) : 0 \leq t_0 < t_1 \cdots < t_n \leq 1 \right\} < \infty.$$

In the work that follows, the semi-martingale concept is extended to processes with two-parameter index set in the plane. The extensions are based on the definitions of bounded variation for real valued functions of two independent variables introduced by Vitali, Arzela and Hardy. These definitions and related concepts can be found in Clarkson and Adams [4].

Motivated by a technique of Föllmer [8] every planar semi-martingale is associated with a measure μ_X . Relationships between μ_X and decompositions of the process X are studied; in particular it is shown that σ -additivity of μ_X is necessary and sufficient for X to possess a Doob-Meyer decomposition. Finally a maximal inequality and almost everywhere convergence theorem is obtained for planar semi-martingales in LlogL. These results are analogs of the linear index theory, however the proofs present new difficulties.

Received March 28, 1979; revised July 16, 1979.

AMS 1970 subject classification: 60G45.

Key words and phrases: Planar semi-martingales, process of bounded variation, normal H -process, maximal inequality, almost everywhere convergence.

* These results represent part of the author's Ph.D. dissertation under the direction of Professor M. M. Rao at the University of California, Riverside. This research was partially supported by Air Force Grant F33615-74-C-4009.

1. BASIC DEFINITIONS AND IMMEDIATE CONSEQUENCES

Let $T = \{(t^1, t^2): 0 \leq t^1 \leq 1, 0 \leq t^2 \leq 1\}$ be the unit square in the plane. For s and t in T , $s < t$ means $s^i \leq t^i$ for $i = 1$ and 2 and $s \ll t$ means $s^i < t^i$ for $i = 1$ and 2 . If $s < t$ in T , then $(s, t] = \{r \in T: s \ll r < t\}$. Such sets in T are called *rectangles*. (The lower bar is used to avoid confusion with intervals of the line $(a, b]$; thus $(s, t] = (s^1, t^1] \times (s^2, t^2]$.) Set $\Gamma_1 = \{(u, 1): 0 \leq u \leq 1\}$, $\Gamma_2 = \{(1, v): 0 \leq v \leq 1\}$, $\Gamma_u = \Gamma_1 \cup \Gamma_2$ and $\Gamma_t = \{(u, v): u = 0 \text{ and } 0 \leq v \leq 1, \text{ or } v = 0 \text{ and } 0 \leq u \leq 1\}$.

If $f: T \rightarrow \mathbb{R}$, the real line, then the *increment of f over $(s, t]$* is given by

$$\Delta f(s, t] = f(t^1, t^2) - f(t^1, s^2) - f(s^1, t^2) + f(s^1, s^2).$$

Note that if $(s, t] = \emptyset$, then $\Delta f(s, t] = 0$ by definition.

If $0 = s_0^i < s_1^i < \dots < s_{n_i}^i = 1$ for $i = 1$ and 2 , then the collection of points $g = \{(s_i^1, s_j^2): 0 \leq i \leq n_1 \text{ and } 0 \leq j \leq n_2\}$ is called a *grid* on T . If $s_{ij} = (s_i^1, s_j^2) \in g$ and $f: T \rightarrow \mathbb{R}$, then $\Delta_g f(s_{ij}) = \Delta f((s_i^1, s_j^2), (s_{i+1}^1, s_{j+1}^2))$.

Let (Ω, \mathcal{F}, P) be a complete probability space. A family of σ -subalgebras of \mathcal{F} , $\{\mathcal{F}(t): t \in T\}$, is an *increasing family* if:

- (1) $s < t$ implies $\mathcal{F}(s) \subseteq \mathcal{F}(t)$,
- (2) $\mathcal{F}(0, 0)$ is P -complete,
- (3) $\mathcal{F}(s) = \bigcap_{s \ll t} \mathcal{F}(t)$.

A family which satisfies (3) is said to be *right order continuous*.

The processes to be defined now will be referred to collectively as *planar semi-martingales* (on T).

DEFINITION 1.1. An adapted process $\{X(t), \mathcal{F}(t), t \in T\}$ contained in $L^1(P)$ is

- (1) a *V-process* with variation K if

$$K = \sup \left\{ E \left(\sum_{t \in g} |E(\Delta_g X(t) | \mathcal{F}(t))| \right) : g \text{ is a grid on } T \right\} < \infty$$

- (2) an *A-process* with variation K if

$$K = \sup \left\{ E \left(\sum_{i=0}^{n-1} |E(X(t_{i+1}) - X(t_i) | \mathcal{F}(t_i))| \right) : \right. \\ \left. (0, 0) < t_0 < t_1 < \dots < t_n < (1, 1) \right\} < \infty$$

(3) an *H-process* if it is a *V-process* and the processes $\{X(t), \mathcal{F}(t), t \in \Gamma_1\}$ and $\{X(t), \mathcal{F}(t), t \in \Gamma_2\}$ are semi-martingales in that for these linear indexes, the supremum of the quantity appearing in (2) above is finite.

If Ω is a one point set, then the above definitions reduce to the respective definitions of bounded variation of Vitali, Arzelà and Hardy (and hence the prefixes *V*, *A*, and *H* used in (1), (2), and (3) respectively).

In extending the martingale and submartingale concepts to the plane, Cairoli [2] studied *V-processes* which satisfy $E(\Delta X(s, t) | \mathcal{F}(s)) \geq 0$ a.e. He called such processes *S-processes* and *M-processes* if equality holds. Later Cairoli and Walsh [3] changed the name of *M-processes* to weak martingales. For the purpose of this paper, such a process will be called a *V-martingale*. More precisely,

DEFINITION 1.2. A *V-process* with variation zero is called a *V-martingale* and an *A-process* with variation zero is called a *planar martingale*.

For $s < t$,

$$\begin{aligned} E(\Delta X(s, t) | \mathcal{F}(s)) &= E(E(X(t^1, t^2) - X(t^1, s^2) | \mathcal{F}(t^1, s^2)) \\ &\quad - (X(s^1, t^2) - X(s^1, s^2)) | \mathcal{F}(s^1, s^2)) \quad \text{a.e.} \end{aligned}$$

which shows that every planar martingale is a *V-martingale*. The inclusion is proper since every process $\{X(t^1, t^2) = V(t^1): (t^1, t^2) \in T\}$ which is a function of one variable is trivially a *V-martingale*, since $\Delta X(s, t) = 0$ for each $s < t$. In general there is no containment relationship between *A-* and *V-processes*. Simple examples can be found in Clarkson and Adams [4] by taking Ω to be a one point set.

In general the three processes of Definition 2.1 are related as follows:

PROPOSITION 1.3. A process $\{X(t), \mathcal{F}(t), t \in T\}$ is an *H-process* if and only if it is a *V-process* and an *A-process*.

Proof. The only nonobvious implication is that an *H-process* is also an *A-process*. Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an *H-process* and $(0, 0) < t_0 < t_1 < \dots < t_n < (1, 1)$ be arbitrary where $t_i = (u_i, v_i)$.

$$\begin{aligned} X(t_{i+1}) - X(t_i) &= -\Delta X(t_i, t_{i+1}] - \Delta X((u_i, v_{i+1}), (u_{i+1}, 1)) \\ &\quad + (X(u_{i+1}, 1) - X(u_i, 1)) - \Delta X((u_{i+1}, v_i), (1, v_{i+1})) \\ &\quad + (X(1, v_{i+1}) - X(1, v_i)). \end{aligned}$$

Thus

$$\begin{aligned} E(|X(t_{i+1}) - X(t_i)| | \mathcal{F}(t_i)) &\leq E(|\Delta X(t_i, t_{i+1}]| | \mathcal{F}(t_i)) \\ &\quad + E(|\Delta X((u_i, v_{i+1}), (u_{i+1}, 1))| | \mathcal{F}(u_i, v_{i+1})) \end{aligned}$$

$$\begin{aligned}
& + E(|E(\Delta X((u_{i+1}, v_i), (1, v_{i+1})) | \mathcal{F}(u_{i+1}, v_i))|) \\
& + E(|E(X(u_{i+1}, 1) - X(u_i, 1) | \mathcal{F}(u_i, 1))|) \\
& + E(|E(X(1, v_{i+1}) - X(1, v_i) | \mathcal{F}(1, v_i))|).
\end{aligned}$$

Summing over i gives

$$E\left(\sum_{i=0}^{n-1} |E(X(t_{i+1}) - X(t_i) | \mathcal{F}(t_i))|\right) \leq 3K + K_1 + K_2 \quad (1.1)$$

where K is the variation of X as a V -process and K_1, K_2 are the variations of the respective semi-martingales $\{X(t), t \in \Gamma_1\}$ and $\{X(t), t \in \Gamma_2\}$. This completes the proof.

A V -martingale is uniquely determined by its values on Γ_u as seen from:

PROPOSITION 1.4. *Let X and Y be two V -martingales. If for each $t \in \Gamma_u$, $X(t) = Y(t)$ a.e., then $X(t) = Y(t)$ a.e. for each $t \in T$. A V -martingale is a planar martingale if and only if $\{X(t), t \in \Gamma_1\}$ and $\{X(t), t \in \Gamma_2\}$ are martingales.*

Proof. Let $t = (t^1, t^2) \in T$, then

$$\begin{aligned}
X(t^1, t^2) &= E(X(1, t^2) + X(t^1, 1) - X(1, 1) | \mathcal{F}(t^1, t^2)) \quad \text{a.e.} \\
&= E(Y(1, t^2) + Y(t^1, 1) - Y(1, 1) | \mathcal{F}(t^1, t^2)) \quad \text{a.e.} \\
&= Y(t^1, t^2) \quad \text{a.e.}
\end{aligned}$$

For the second assertion, necessity is obvious and sufficiency follows from equation (1.1) with $K = K_1 = K_2 = 0$.

The limit behavior of V -processes can be quite bad; for example, $X(t^1, t^2) = Y(t^1)$ is such that $\Delta X(s, t) = 0$, and hence $\{X(t), t \in T\}$ is a V -martingale regardless of the regularity (in any sense) of $Y(\cdot)$. This is not true of A -processes.

THEOREM 1.5. *Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an A -process with variation K , then $\lim_{s \rightarrow t, s \gg t} X(s)$ exists in L^1 -mean for $t \in T - \Gamma_u$, and the process*

$$Y(t) = \begin{cases} L^1 - \lim_{s \rightarrow t, s \gg t} X(s) & \text{for } t \in T - \Gamma_u \\ X(1, 1) & \text{for } t = (1, 1) \\ \lim_{h \rightarrow 0+} X(t^1 + h, 1) & \text{for } t \in \Gamma_1 - (1, 1) \\ \lim_{h \rightarrow 0+} X(1, t^2 + h) & \text{for } t \in \Gamma_2 - (1, 1) \end{cases}$$

is an A -process with variation K . Moreover, $\lim_{s \rightarrow t, s \ll t} X(s)$ exists in probability for $t \in T - \Gamma_l$.

Proof. Suppose for some $t \in T - \Gamma_u$ that $\lim_{s \rightarrow t} X(s)$ in L^1 does not exist, then the generalized sequence $\{X(s): s \gg t\}$ cannot be L^1 -Cauchy. Hence there exists $\epsilon > 0$ such that for each $s \gg t$, there are p and q satisfying $s \gg p \gg t$ and $s \gg q \gg t$ such that $\|X(p) - X(q)\|_1 \geq \epsilon$. By the triangle inequality, $\max\{\|X(s) - X(p)\|_1, \|X(s) - X(q)\|_1\} \geq \epsilon/2$. This allows the construction of a sequence $\{X(t_i): t_i \gg t_{i+1} \gg t\}$ such that $\|X(t_i) - X(t_{i+1})\|_1 \geq \epsilon/2$. By Theorem 2.2 of Orey [10], $\lim_i X(t_i)$ exist a.e. $[P]$. By Theorem 2.3 of the same paper, $\{X(t_i)\}$ is a uniformly integrable sequence and so must be L^1 -convergent by the Vitali convergence theorem. This contradiction proves the first assertion.

Let $t_0 < t_1 < t_2 < \dots < t_n$ be arbitrary points in T . Set

$$a_i^k = \begin{cases} (1/k, 1/k) & \text{if } t_i \in T - \Gamma_u \\ (0, 0) & \text{if } t_i = (1, 1) \\ (0, 1/k) & \text{if } t_i \in \Gamma_2 - (1, 1) \\ (1/k, 0) & \text{if } t_i \in \Gamma_1 - (1, 1). \end{cases}$$

Then

$$\begin{aligned} & E \left(\sum_{i=0}^{n-1} |E(Y(t_{i+1}) - Y(t_i) | \mathcal{F}(t_i))| \right) \\ &= \lim_{k \rightarrow \infty} E \left(\sum_{i=0}^{n-1} |E(X(t_{i+1} + a_{i+1}^k) - X(t_i + a_i^k) | \mathcal{F}(t_i + a_i^k))| \right) \leq K. \quad (1.2) \end{aligned}$$

The inequality is from the definition of K . The equality is seen by examining a typical term.

$$\begin{aligned} & \|E(X(t_{i+1} + a_{i+1}^k) | \mathcal{F}(t_i + a_i^k)) - E(Y(t_{i+1}) | \mathcal{F}(t_i))\|_1 \\ & \leq \|E(X(t_{i+1} + a_{i+1}^k) | \mathcal{F}(t_i + a_i^k)) - E(Y(t_{i+1}) | \mathcal{F}(t_i + a_i^k))\|_1 \\ & \quad + \|E(Y(t_{i+1}) | \mathcal{F}(t_i + a_i^k)) - E(Y(t_{i+1}) | \mathcal{F}(t_i))\|_1 \\ & \leq \|X(t_{i+1} + a_{i+1}^k) - Y(t_{i+1})\|_1 \\ & \quad + \|E(Y(t_{i+1}) | \mathcal{F}(t_i + a_i^k)) - E(Y(t_{i+1}) | \mathcal{F}(t_i))\|_1. \end{aligned}$$

As $k \rightarrow \infty$, the first term goes to zero by the definition of $Y(t_{i+1})$ and the second term goes to zero by the martingale convergence theorem and the right order continuity of $\{\mathcal{F}(t), t \in T\}$.

If for some $t \in T - \Gamma_l$, $\lim_{s \rightarrow t} X(s)$ does not exist in probability, then the generalized sequence $\{X(s): s \ll t\}$ cannot be Cauchy in the metric $d(\cdot)$, where $d(X) = \int_{\Omega} [|X|/(1 + |X|)] dP$. The same argument as above produces a sequence $\{X(t_i)\}$ such that $d(X(t_{i+1}) - X(t_i)) \geq \epsilon/2$, but $\lim_{i \rightarrow \infty} X(t_i)$ exists a.e. This contradiction proves the last assertion and completes the proof.

Theorem 1.5 remains true if X is moreover an H -process. A calculation similar to (1.2) shows that Y as a V -process has the same variation as X .

The following subsets of V and A -processes are of interest:

DEFINITION 1.6. A V -submartingale is a process $\{X(t), \mathcal{F}(t), t \in T\}$ satisfying for each $s < t$ in T , $E(\Delta X(s, t) | \mathcal{F}(s)) \geq 0$ a.e. An A -submartingale is a process satisfying $E(X(t) | \mathcal{F}(s)) \geq E(X(s))$ a.e. for $s < t$ in T . If the inequalities are reversed, the process will be called a V -supermartingale (respectively A -supermartingale).

2. MEASURE REPRESENTATIONS

This section will detail the generation of certain measures induced by V -processes for the purpose of developing decomposition theorems with the aid of classical decompositions of measure theory. These techniques were developed in the case of linear indexed processes by Doleans-Dade [5] for supermartingales and Föllmer [8] for semi-martingales.

Setting $T_0 = T - \Gamma_l$, the following classes of subsets of $T_0 \times \Omega$ will be of interest:

$\mathcal{R} = \{(s, t] \times A : s \ll t \text{ and } A \in \mathcal{F}_s\}$. Elements of \mathcal{R} are called *predictable rectangles*; \mathcal{R} is a semi-algebra.

\mathcal{O} = the smallest algebra generated by \mathcal{R} ; that is, finite disjoint unions of predictable rectangles.

\mathcal{P} = the σ -algebra generated by \mathcal{R} . It can be shown that \mathcal{P} is also generated by all left-order continuous processes of $T \times \Omega$.

With any process $X(\cdot, \cdot) : T \times \Omega \rightarrow \mathbb{R}$, adapted to $\{\mathcal{F}(t), t \in T\}$ such that $X(t, \cdot) \in L^1(\Omega, \mathcal{F}, P)$, define $\mu_X : \mathcal{R} \rightarrow \mathbb{R}$ by

$$\mu_X((s, t] \times A) = \int_A \Delta X(s, t) dP \quad (2.1)$$

If $(s, t] \times A = \bigcup_{i=1}^n (s_i, t_i] \times A_i$ (disjoint union), then denote $I(\omega) = \{i : \omega \in A_i\}$. If $\omega \in A$, then $(s, t] = \bigcup \{(s_i, t_i] : i \in I(\omega)\}$. In other words

$$\chi_A(\omega) \Delta X(s, t] = \sum_{i=1}^n \chi_{A_i}(\omega) \Delta X(s_i, t_i].$$

Integrating both sides with respect to P gives

$$\mu_X((s, t] \times A) = \sum_{i=1}^n \mu_X((s_i, t_i] \times A_i),$$

showing that μ_X is finite and finitely additive on \mathcal{R} ; by a standard measure theory argument, $\mu_X(\cdot)$ has a unique finitely additive extension to \mathcal{O} . The extension will also be denoted by the same symbol, μ_X .

If $\mu: \mathcal{O} \rightarrow \mathbb{R}$ is a finitely additive set function, let $|\mu|$ denote the total variation of μ . V -processes are distinguished by:

PROPOSITION 2.1. *If $X: T \times \Omega \rightarrow \mathbb{R}$ is adapted to $\{\mathcal{F}(t), t \in T\}$ and contained in $L^1(P)$, then X is a V -process if and only if $|\mu_X|(T_0 \times \Omega) < \infty$ and X has variation equal to $|\mu_X|$.*

Proof. Let $g \subset T$ be any grid and for $t \in g$ set $A_t = \{\omega: E(\Delta_g X(t) | \mathcal{F}(t))(\omega) \geq 0\}$.

$$\begin{aligned} E\left(\sum_{t \in g} |E(\Delta_g X(t) | \mathcal{F}(t))|\right) \\ &= E\left(\sum_{t \in g} (\Delta_g X(t))(\chi_{A_t} - \chi_{A_t^c})\right) \\ &\leq \sum_{t \in g} \left(\left| \int_{A_t} \Delta_g X(t) dP \right| + \left| \int_{A_t^c} \Delta_g X(t) dP \right| \right) \\ &\leq \sup \left\{ \sum_{t \in g} \left| \int_{A_t} \Delta X(\underline{s}_t, \underline{t}_t) dP \right| : \bigcup_{t \in g} A_t \times (\underline{s}_t, \underline{t}_t) \in \mathcal{O} \right\} = |\mu_X|(T_0 \times \Omega), \end{aligned}$$

which shows that the variation of X is $\leq |\mu_X|$. For the converse, let $A = \bigcup_{i=1}^n (\underline{s}_i, \underline{t}_i] \times B_i$ belong to \mathcal{O} . By taking some B_i 's empty if necessary, assume $T_0 = \bigcup_{i=1}^n (\underline{s}_i, \underline{t}_i]$. Let $0 = u_0 < u_1 < u_2 < \dots < u_{m_1} = 1$ be a set of real numbers which contains all the first coordinates of \underline{s}_i and \underline{t}_i for $1 \leq i \leq n$, and $0 = v_0 < v_1 < \dots < v_{m_2}$ contain all the second coordinates. The grid $r_{jk} = (u_j, v_k)$ is such that $(\underline{s}_i, \underline{t}_i] \cap (\underline{r}_{jk}, \underline{r}_{j+1, k+1}] = (\underline{r}_{jk}, \underline{r}_{j+1, k+1}]$ or \emptyset . Set $C_{jk} = \bigcup \{B_i: \text{the above intersection is nonempty}\}$. The above construction is such that

$$A = \bigcup_{i=1}^n (\underline{s}_i, \underline{t}_i] \times B_i = \bigcup_{j,k} (\underline{r}_{jk}, \underline{r}_{j+1, k+1}] \times C_{jk}$$

and the second representation refines the first.

$$\begin{aligned} \sum_{i=1}^n |\mu_X((\underline{s}_i, \underline{t}_i] \times B_i)| &\leq \sum_{j,k} |\mu_X((\underline{r}_{jk}, \underline{r}_{j+1, k+1}] \times C_{jk})| \\ &\leq \sum_{j,k} \int_{C_{jk}} |E(\Delta X(r_{jk}) | \mathcal{F}(r_{jk}))| dP \\ &\leq E\left(\sum_{j,k} |E(\Delta X(r_{jk}) | \mathcal{F}(r_{jk}))|\right), \end{aligned}$$

which shows $|\mu_X|$ is less than or equal to the variation of X .

Let X be a V -process on T ; consider the map $Q_t: \mathcal{F}_t \rightarrow \mathbb{R}$ defined by

$$Q_t(A) = \mu_X(\underline{(t, (1, 1)]} \times A) = \int_A \Delta X(t, \underline{(1, 1)}) dP. \quad (2.2)$$

Q_t is countably additive and P -continuous. Define

$$\bar{X}(t) = \frac{dQ_t}{dP}. \quad (2.3)$$

Now,

$$\begin{aligned} \mu_X(\underline{(s, t]} \times A) &= \mu_X(\underline{(s, (1, 1)]} \times A) - \mu_X(\underline{((s^1, t^2), (1, 1)]} \times A) \\ &\quad - \mu_X(\underline{((t^1, s^2), (1, 1)]} \times A) + \mu_X(\underline{(t, (1, 1)]} \times A) \\ &= \int_A \Delta \bar{X}(s, t) dP = \mu_{\bar{X}}(\underline{(s, t]} \times A). \end{aligned}$$

Since $\mu_X = \mu_{\bar{X}}$, \bar{X} is a V -process; also $\bar{X}(t) = 0$ a.e. for $t \in \Gamma_u$, so \bar{X} is in fact an H -process.

DEFINITION 2.2. A V -process X on T such that $X(t) = 0$ a.e. for $t \in \Gamma_u$ is called a *normal H -process*.

The next proposition is an immediate consequence of the construction given by (2.2) and (2.3).

PROPOSITION 2.3. For each finitely additive measure $\mu: \mathcal{O} \rightarrow \mathbb{R}$, such that the map $A \mapsto \mu(\underline{(t, (1, 1)]} \times A)$ is countably additive from $\mathcal{F}(t)$ to \mathbb{R} and P -continuous, there exists a unique normal H -process X such that $\mu = \mu_X$.

Suppose X is a normal H -process and μ_X is a positive measure, then X has the additional properties:

- (1) $X(t) \geq 0$ a.e. for each $t \in T$
 - (2) X is a V -submartingale
 - (3) X is an A -supermartingale
- (2.4)

All but (3) are obvious. If $s < t$ in T and $A \in \mathcal{F}(s)$,

$$\begin{aligned} \int_A (X(s) - X(t)) dP &= \int_A X(s) dP - \int_A X(t) dP \\ &= \mu_X(\underline{(s, (1, 1)]} \times A) - \mu_X(\underline{(t, (1, 1)]} \times A) \geq 0 \end{aligned}$$

since $\underline{(t, (1, 1)]} \times A \subseteq \underline{(s, (1, 1)]} \times A$ and μ_X is positive.

Classical measure theory and the above results lead to the following theorem.

THEOREM 2.4. *Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an L^1 -right continuous V -process, then X decomposes as $X = M + Y - Z$ where M is an L^1 -right continuous V -martingale and Y and Z are L^1 -right continuous normal H -processes satisfying the conditions (2.4).*

Proof. Let \bar{X} be the normal H -process such that $\mu_X = \mu_{\bar{X}}$ given by Proposition 2.3. For $s \in T - \Gamma_u$, $\bar{X}(s+) = L^1 - \lim_{t \rightarrow s, t \gg s} \bar{X}(s)$ exists. For any $A \in \mathcal{F}(s)$,

$$\begin{aligned} \int_A \bar{X}(s) dP &= \int_A \Delta X(s, (1, 1)] dP \\ &= \lim_{t \gg s} \int_A \Delta X(t, (1, 1)] dP, \quad \text{by the } L^1\text{-convergence,} \\ &= \lim_{t \gg s} \int_A \bar{X}(t) dP \\ &= \int_A \bar{X}(s+) dP, \quad \text{by the } L^1\text{-convergence again,} \end{aligned}$$

which implies $\bar{X}(s) = \bar{X}(s+)$ a.e. and \bar{X} is L^1 -right continuous. Set $M = X - \bar{X}$; since $\mu_M = 0$, M is an L^1 -right continuous V -martingale.

Define $Q_s^n: \mathcal{F}(s) \rightarrow \mathbb{R}$ by

$$\begin{aligned} Q_s^n(A) &= \sup \left\{ \sum_{i,j=0}^{2^n-1} \left| \int_{A_{ij}} \Delta X\left(\frac{i}{2^n}, \frac{j}{2^n}\right) dP \right| : A_{ij} \in \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right), \right. \\ &\quad \left. A_{ij} \subseteq A, A_{ij} = \emptyset \text{ unless } s < \left(\frac{i}{2^n}, \frac{j}{2^n}\right) \right\}. \end{aligned}$$

It is easily seen that Q_s^n is a countably additive P -continuous measure on $\mathcal{F}(s)$. Moreover, $Q_s^n(A) \leq Q_s^{n+1}(A) \leq |\mu_X|((s, (1, 1)] \times A)$ for each $A \in \mathcal{F}(s)$. Hence, $Q_s(A) = \lim_{n \rightarrow \infty} Q_s^n(A)$ exists for each $A \in \mathcal{F}(s)$. By the Vitali-Hahn-Saks theorem (cf. Dunford and Schwartz [7, p. 158–160]), Q_s is a countably additive P -continuous measure on the σ -algebra $\mathcal{F}(s)$.

For $s \in T$, $A \in \mathcal{F}(s)$, and $\epsilon > 0$, let $\{r_{ij}\}$ be a grid and $A_{ij} \in \mathcal{F}(r_{ij})$ with $A_{ij} \subseteq A$ and $A_{ij} = \emptyset$ unless $s < r_{ij}$, such that

$$|\mu_X|((s, (1, 1)] \times A) \leq \sum_{i,j} \left| \int_{A_{ij}} \Delta X(r_{ij}) dP \right| + \epsilon. \quad (2.5)$$

Set $r_{ij}^n = (k/2^n, l/2^n)$ if $r_{ij} \in (((k-1)/2^n, (l-1)/2^n), (k/2^n, l/2^n)]$. By the L^1 -right continuity of X , there exists n_0 such that $n \geq n_0$ implies

$$\sum_{i,j} \left| \int_{A_{ij}} \Delta X(r_{ij}) dP \right| \leq \sum_{i,j} \left| \int_{A_{ij}} \Delta X(r_{ij}^n) dP \right| + \epsilon \leq Q_s^n(A) + \epsilon. \quad (2.6)$$

Substituting (2.6) into (2.5) gives $|\mu_X|((s, (1, 1)] \times A) \leq Q_s^n(A) + 2\epsilon$ for $n \geq n_0$. Since ϵ is arbitrary, this implies

$$|\mu_X|((s, (1, 1)] \times A) = Q_s(A) \quad (2.7)$$

for each $A \in \mathcal{F}(s)$.

Set $\mu^+ = \frac{1}{2}(|\mu_X| + \mu_X)$ and $\mu^- = \frac{1}{2}(|\mu_X| - \mu_X)$. This is the classical Jordan decomposition of μ_X . By (2.7) μ^+ and μ^- satisfy the hypothesis of Proposition 2.3, so there exists normal H -processes Y and Z which satisfy (2.4) and are such that $\mu^+ = \mu_Y$ and $\mu^- = \mu_Z$. For each $s \in T$ and $A \in \mathcal{F}(s)$,

$$\begin{aligned} \int_A \bar{X}(s) dP &= \mu_X((s, (1, 1)] \times A) = \mu_X((s, (1, 1)] \times A) \\ &= (\mu^+ - \mu^-)((s, (1, 1)] \times A) = \int_A (Y(s) - Z(s)) dP. \end{aligned}$$

So $\bar{X}(s) = Y(s) - Z(s)$ a.e. or $X = M + Y - Z$. Y and Z are L^1 -right continuous since the map $t \mapsto |\mu_X|((t, (1, 1)] \times A)$ is right continuous by the L^1 -right continuity of X . This completes the proof.

Let K_X, K_Y, K_Z be the variations of X, Y , and Z as V -processes. From Proposition 2.1 and the fact that $\mu_Y = \mu^+$ and $\mu_Z = \mu^-$, $K_X = K_Y + K_Z$. Y and Z are the only normal H -processes satisfying (2.4) with this property. For if $X = M + Y' - Z'$, then $\mu_X = \mu_{Y'} - \mu_{Z'}$; but $|\mu_X| \leq \mu_{Y'} + \mu_{Z'}$ with equality only if $\mu_{Y'} = \mu^+$ and $\mu_{Z'} = \mu^-$.

3. σ -ADDITIVITY AND DOOB-MEYER DECOMPOSITIONS

For an L^1 -right continuous process $\{X(t), \mathcal{F}(t), t \in T\}$ set $I_X = \{\sum_{i,j=0}^{2^n-1} |E(\Delta X(i/2^n, j/2^n) | \mathcal{F}(i/2^n, j/2^n))| : n \geq 1\}$. By Proposition 2.1 and the L^1 -right continuity of X , μ_X is of finite variation if and only if I_X is contained in a ball of $L^1(P)$. The work that follows will investigate further relationships between X , μ_X and I_X .

If $A \subset T \times \Omega$, set $\pi(A) = \{\omega : (t, \omega) \in A \text{ for some } t \in T\}$.

DEFINITION 3.1. A measure $\mu: \mathcal{O} \rightarrow \mathbb{R}$ of finite variation is *projection continuous* if given $\epsilon > 0$, there exists $\delta > 0$, depending only on ϵ , such that if $A \in \mathcal{O}$ satisfies $P\{\pi(A)\} < \delta$, then $|\mu|(A) < \epsilon$.

If X is a V -process, μ_X is projection continuous if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $g \subset T$ is any grid and $\{A(t) \in \mathcal{F}(t) : t \in g\}$ is a collection of sets satisfying $P\{\bigcup_{t \in g} A(t)\} < \delta$, then $\sum_{t \in g} |\int_{A(t)} \Delta_g X(t) dP| < \epsilon$. This is easily seen if one recalls that if $A \in \mathcal{O}$, there exists a grid $\{r_{ij}\}$ such that $A = \bigcup_{ij} \underline{(r_{ij}, r_{i+1,j+1})} \times B_{ij}$. This was shown in the proof of Proposition 2.1.

Two examples of processes with projection continuous measures are

(1) any V -martingale and (2) any process $\{X(t), \mathcal{F}(t), t \in T\}$ which is positive, a V -submartingale and an A -submartingale. To see the truth for the second example, let $\{t_{ij} = (u_i, v_j)\}$ be a grid and $A_{ij} \in \mathcal{F}(t_{ij})$. Set $B_{i0} = A_{i0}$, $B_{ij} = A_{ij} - \bigcup_{k=0}^{j-1} A_{ik}$ for $j > 0$, $C_i = \bigcup_j A_{ij}$, $D_0 = C_0$ and $D_i = C_i - \bigcup_{k=0}^{i-1} C_k$ for $i > 0$.

$$\begin{aligned} & \sum_{ij} \left| \int_{A_{ij}} \Delta X(t_{ij}) dP \right| \\ &= \sum_{ij} \int_{A_{ij}} \Delta X(t_{ij}) dP \\ &= \sum_{ij} \int_{B_{ij}} (X(u_{i+1}, 1) - X(u_i, 1) - X(u_{i+1}, v_j) + X(u_i, v_j)) dP \\ &\leq \sum_{i,j} \int_{B_{ij}} (X(u_{i+1}, 1) - X(u_i, 1)) dP = \sum_i \int_{C_i} (X(u_{i+1}, 1) - X(u_i, 1)) dP \\ &= \sum_i \int_{D_i} (X(1, 1) - X(u_i, 1)) dP = \int_{\bigcup D_i} X(1, 1) dP \\ &= \int_{\bigcup A_{ij}} X(1, 1) dP \rightarrow 0 \end{aligned}$$

as $P(\bigcup A_{ij}) \rightarrow 0$. These examples seem rather simple, but later results will show that they are essentially the only examples.

For any map $A: T \times \Omega \rightarrow \mathbb{R}$, set $\|A\| = \sup\{\sum_{t \in g} |\Delta_g A(t)| : g \subset T, \text{ a grid}\}$. $\|A\|(\cdot)$ maps Ω into $\mathbb{R}^+ \cup \{+\infty\}$.

DEFINITION 3.2. $\{A(t), \mathcal{F}(t), t \in T\}$ is a *process of bounded variation* if

- (1) $A(t) = 0$ a.e. for each $t \in \Gamma_t$;
- (2) The map $t \mapsto A(t, \omega)$ is right continuous for almost all ω ;
- (3) $\|A\| \in L^1(P)$.

If (1) and (3) hold then for almost all ω the map $t \mapsto A(t, \omega)$ is of bounded variation in the sense of Hardy.

Let $\{A(t), \mathcal{F}(t), t \in T\}$ be a process of bounded variation. For $C \times (s, t]$, a rectangle in $\mathcal{F} \otimes \mathcal{B}(T)$, where $\mathcal{B}(T)$ is the Borel σ -algebra of T , set $\mu_A(C \times (s, t]) = \int_C \Delta A(s, t] dP$. Note that C need not be $\mathcal{F}(s)$ -measurable. A version of Fubini's theorem (cf. Rao [12, Chap. II]) states that μ_A has a unique countably additive extension to $\mathcal{F} \otimes \mathcal{B}(T)$ and that for $f \in L^1(\mu_A)$,

$$\int_{T \times \Omega} f d\mu_A = E \left(\int_T f(t, \cdot) dA(t) \right)$$

where the extension is also denote by μ_A .

Consider μ_A restricted to \mathcal{O} . By Proposition 2.3 there exists a unique normal H -process $\{X(t), \mathcal{F}(t), t \in T\}$ such that $\mu_A|_{\mathcal{O}} = \mu_X$. Now, two results are evident. One, μ_X has a countably additive extension to \mathcal{P} namely $\mu_A|_{\mathcal{P}}$, and two, since $\mu_A = \mu_X$ on \mathcal{O} , for $B \in \mathcal{F}(s)$,

$$\int_B X(s) dP = \mu_X((s, (1, 1)] \times B) = \mu_A((s, (1, 1)] \times B) = \int_B \Delta A(s, (1, 1)] dP$$

or

$$X(s) = E(\Delta A(s, (1, 1)] | \mathcal{F}(s)) \quad \text{a.e.} \quad (3.1)$$

DEFINITION 3.3. If $\{A(t), \mathcal{F}(t), t \in T\}$ is a process of bounded variation, $\{X(t), \mathcal{F}(t), t \in T\}$ defined by (3.1) is called the *normal H -process generated by A* .

Let $\{X(t), \mathcal{F}(t), t \in [0, 1]\}$ be a right continuous semi-martingale, then $X = M + B$ where M is a martingale, B has paths of bounded variation and the map $t \mapsto B(t)$ is of L^1 -bounded variation if and only if X belongs to *class D* . X is said to be of class D if the family $\{X \cdot \sigma: \sigma \text{ a stopping time of } \{\mathcal{F}(t)\}\}$ is uniformly integrable. (A stopping time is a map $\sigma: \Omega \rightarrow [0, 1]$ such that the set $\{\sigma \leq t\} \in \mathcal{F}(t)$ for each $t \in [0, 1]$.) This is the classical continuous time Doob-Meyer decomposition. Details and generalizations can be found in Rao [12, Chap. IV]. Although defining class D for V -processes and stopping times with values in T would make sense, membership in such a class would not be necessary and sufficient for a planar Doob-Meyer decomposition since there would exist V -martingales not in such a planar class D . The next condition will be necessary and sufficient for a planar Doob-Meyer decomposition and since it is also a condition on the uniform integrability of X will be called class D' .

DEFINITION 3.4. A V -process X belongs to *class D'* if the set I_X is uniformly integrable.

The next theorem unifies the above concepts. Part (5) may be regarded as a Doob-Meyer decomposition. The essential idea in showing (3) implies (4) can be found in Cairoli [2].

THEOREM 3.5. Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an L^1 -right continuous V -process with associated normal H -process $\{\bar{X}(t), \mathcal{F}(t), t \in T\}$. The following are equivalent:

- (1) X belongs to class D' .
- (2) μ_X is projection continuous.
- (3) μ_X has a unique countably additive extension to \mathcal{P} .
- (4) There exists a process of bounded variation which generates \bar{X} .

(5) There exists a decomposition $X = M + A$ where M is a V -martingale and A is a process of bounded variation, each adapted to $\{\mathcal{F}(t), t \in T\}$.

Proof. Suppose X belongs to class D' and let B_{ij} belong to $\mathcal{F}(i/2^n, j/2^n)$.

$$\sum_{ij} \left| \int_{B_{ij}} \Delta X(i/2^n, j/2^n) dP \right| \leq \int_{\bigcup B_{ij}} \sum_{ij} |E(\Delta X(i/2^n, j/2^n) | \mathcal{F}(i/2^n, j/2^n))| dP$$

which goes to zero uniformly in n as $P\{\bigcup B_{ij}\} \rightarrow 0$ since X belongs to class D' . This shows that μ_X is projection continuous on elements of \mathcal{O} constructed on dyadic grids. The general case then follows by the L^1 -right continuity of X .

Suppose (2) is true; to show (3) holds, it suffices to show if $B_n \downarrow \emptyset$, $B_n \in \mathcal{O}$, then $\mu_X(B_n) \rightarrow 0$. By Theorem 2.4 and Definition 3.1, there is no loss of generality in assuming $\mu_X \geq 0$. Choose and fix $\epsilon > 0$, $B_n = \bigcup_{i=1}^{m_n} (s_i^n, t_i^n] \times B_i^n$. Set $K_n = \bigcup_{i=1}^{m_n} (r_i^n, t_i^n] \times B_i^n \in \mathcal{O}$, where each r_i^n is chosen so that $s_i^n \ll r_i^n \ll t_i^n$ and $\mu_X(K_n) + \epsilon/2^n > \mu_X(B_n)$. This can be done by the L^1 -right continuity of X . Set $L_n = \bigcap_{j=1}^n K_j$, then $L_n \downarrow \emptyset$ and $\mu_X(B_n) \leq \mu_X(L_n) + \epsilon$. For $A \subset T$, denote by $\text{cl}(A)$ the closure of A in the standard (Euclidean) topology of T . Suppose $\omega_0 \in \bigcap_n \pi(L_n)$, then $L_n^{\omega_0} = \{t: (t, \omega_0) \in L_n\} \neq \emptyset$. By the finite intersection property of the compact sequence $\{\text{cl}(L_n^{\omega_0})\}$, there exists a $t_0 \in \bigcap_n \text{cl}(L_n^{\omega_0})$. By the construction $\text{cl}(L_n^{\omega_0}) \subseteq \text{cl}(K_n^{\omega_0}) \subseteq B_n^{\omega_0}$, which implies $(t_0, \omega_0) \in B_n$ for each n . This contradicts the hypothesis that $B_n \downarrow \emptyset$; thus $\bigcap_n \pi(L_n) = \emptyset$ and $\lim_{n \rightarrow \infty} P(\pi(L_n)) = 0$. Since μ_X is projection continuous, $\mu_X(L_n) \rightarrow 0$. Hence $\lim_{n \rightarrow \infty} \mu_X(B_n) \leq \epsilon$, but ϵ was arbitrary, so $\lim_{n \rightarrow \infty} \mu_X(B_n) = 0$. This proves that μ_X is countably additive.

Assume that (3) is true. For $s \in T$ set $s^n = (i/2^n, j/2^n)$ if $s \in ((i/2^n, j/2^n), ((i+1)/2^n, (j+1)/2^n)]$. For $B \in \mathcal{F}$ set $Y_B(s, \omega) = \lim_{n \rightarrow \infty} E(\chi_B | \mathcal{F}(s^n))(\omega)$. This limit exists and in fact equals $E(\chi_B | \mathcal{F}(s-))$ a.e., by the martingale convergence theorem. $(\mathcal{F}(s)- = \bigvee_{t \ll s} \mathcal{F}(t))$. Y_B as a map from $T \times \Omega$ to \mathbb{R} is \mathcal{P} measurable.

For each $t = (t^1, t^2) \in T$, consider the map $\sigma_t: \mathcal{F}_t \rightarrow \mathbb{R}$ defined by

$$\sigma_t(B) = \int_{((0,0),t] \times \Omega} Y_B d\mu_X. \quad (3.2)$$

By construction

$$Y_B(s, \omega) = \lim_{n \rightarrow \infty} \sum_{i,j=0}^{2^n-1} E\left(\chi_B \mid \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right)(\omega) \chi_{((i/2^n, j/2^n), ((i+1)/2^n, (j+1)/2^n))}(s).$$

Now, the bounded convergence theorem (σ -additivity of μ_X is used here) implies

$$\begin{aligned} \sigma_t(B) &= \lim_{n \rightarrow \infty} \sum_{i,j=0}^{2^n-1} \int_{\Omega} E\left(\chi_B \mid \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right) \\ &\quad \times \Delta \bar{X}\left(\left(\frac{i}{2^n}, \frac{j}{2^n}\right) \wedge t, \left(\frac{i+1}{2^n}, \frac{j+1}{2^n}\right) \wedge t\right] dP. \end{aligned} \quad (3.3)$$

Set

$$A^n(t) = \sum_{i,j=0}^{2^n-1} E \left(\Delta \bar{X} \left(\left(\frac{i}{2^n}, \frac{j}{2^n} \right) \wedge t, \left(\frac{i+1}{2^n}, \frac{j+1}{2^n} \right) \wedge t \right) \middle| \mathcal{F} \left(\frac{i}{2^n}, \frac{j}{2^n} \right) \right).$$

Using the self-adjoint property of conditional expectations and substituting $A^n(t)$ into (3.3) gives

$$\sigma_t(B) = \lim_{n \rightarrow \infty} E(\chi_B A^n(t)). \quad (3.4)$$

By the Vitali-Hahn-Saks theorem, $\sigma_t: \mathcal{F}_t \rightarrow \mathbb{R}$ is countably additive and there exists an a.e. unique $\mathcal{F}(t)$ -measurable function $\bar{A}(t)$ such that

$$\sigma_t(B) = E(\chi_B \bar{A}(t)) = \int_{(\underline{0}, \underline{0}], t] \times \Omega} Y_B d\mu_X, \quad (3.5)$$

holds for all $B \in \mathcal{F}(t)$.

Set $A(t) = \lim_{s \gg t} \bar{A}(s)$, by the σ -additivity of μ_X , this limit exists and (3.5) holds true with \bar{A} replaced by A . A is only a modification of \bar{A} in the sense that for each t , $A(t) = \bar{A}(t)$ a.e.

From (3.5) it follows that

$$\int_B \Delta A(\underline{s}, \underline{t}] dP = \int_{(\underline{s}, \underline{t}] \times \Omega} Y_B d\mu_X \quad (3.6)$$

for all $B \in \mathcal{F}(t)$.

Suppose that μ_X is positive; from (3.5) and (3.6) it follows that

- (1) $A(t) = 0$ a.e. for $t \in \Gamma_t$
- (2) $E(A(1, 1)) = \mu_X(T_0 \times \Omega) < \infty$
- (3) $\Delta A(\underline{s}, \underline{t}] \geq 0$ for all $s < t$ in T .

If $B \in \mathcal{F}(s)$ in (3.5), then (3.5) reduces to

$$\begin{aligned} \int_B \Delta A(\underline{s}, \underline{t}] dP &= \int_{(\underline{s}, \underline{t}] \times \Omega} \chi_B d\mu_X = \int_{(\underline{s}, \underline{t}] \times \Omega} \chi_B d\mu_{\bar{X}} \\ &= \mu_{\bar{X}}((\underline{s}, \underline{t}] \times B) = \int_B \Delta \bar{X}(\underline{s}, \underline{t}] dP. \end{aligned}$$

Since this holds for all $B \in \mathcal{F}(s)$,

$$E(\Delta A(\underline{s}, (1, 1]) \mid \mathcal{F}(s)) = \bar{X}(s) \quad \text{a.e.} \quad (3.7)$$

A is a process of bounded variation which generates \bar{X} . This completes the proof that (3) implies (4) except for remarking that the case μ_X is a signed measure is treated as above by considering each part of its Jordan decomposition.

Suppose (4) holds, so that

$$\begin{aligned} X(s) &= E(\Delta A(s, (1, 1)) \mid \mathcal{F}(s)) \\ &= A(s) + E(A(1, 1) - A(1, s^2) - A(s^1, 1) \mid \mathcal{F}(s)) \\ &= A(s) + M_1(s), \end{aligned}$$

where M_1 is a V -martingale. Since $\mu_X = \mu_A$, $X = M + \bar{X}$, where M is a V -martingale. Thus, $X = (M + M_1) + A$, proving (5).

(5) implies (3) immediately since $\mu_X = \mu_A$. (3) implies (1) is obtained by replacing t by $(1, 1)$ in equations (3.4) and (3.5). This shows that $\bar{A}(1, 1) = \lim_{n \rightarrow \infty} I_X$ where the limit is in the weak topology of $L^1(P)$. I_X must be uniformly integrable by Theorem 9, page 292, of Dunford and Schwartz [7]. This completes the proof of the theorem.

A process $\{X(t), \mathcal{F}(t), t \in T\}$ that is both a V -submartingale and an A -submartingale will be called an H -submartingale. H -submartingales are characterized by:

THEOREM 3.6. *Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an L^1 -right continuous H -process, then the following are equivalent:*

(1) X belongs to class D' and $\{X(t), t \in \Gamma_1\}$ and $\{X(t), t \in \Gamma_2\}$ each belong to class D .

(2) $X = Y - Z$ where Y and Z are positive L^1 -right continuous H -submartingales.

Proof. Let (1) hold, by Theorem 3.5 $X = M + A$ where M is a V -martingale and A is a process of bounded variation. $A = A^+ - A^-$ where A^+ and A^- are the positive and negative variations of A .

$$\begin{aligned} M(u, v) &= E(M(1, v) \mid \mathcal{F}(u, v)) + E(M(u, 1) \mid \mathcal{F}(u, v)) \\ &\quad - E(M(1, 1) \mid \mathcal{F}(u, v)). \end{aligned} \quad (3.8)$$

Consider the first term. $\{M(1, v): 0 \leq v \leq 1\}$ is a semi-martingale in class D since $\{X(t), t \in \Gamma_2\}$ and $\{A(t), t \in \Gamma_2\}$ each belong to class D . Hence $M(1, v) = \bar{M}(v) + B^+(v) - B^-(v)$ where $\{\bar{M}(v), \mathcal{F}(1, v), v \in [0, 1]\}$ is a right continuous martingale and $\{B^+(v), \mathcal{F}(1, v), v \in [0, 1]\}$ is a right continuous process with increasing paths such that $E(B^+(1)) < \infty$. The same holds for B^- .

$$\begin{aligned} E(M(1, v) \mid \mathcal{F}(u, v)) &= E(\bar{M}(v) \mid \mathcal{F}(u, v)) + E(B^+(v) \mid \mathcal{F}(u, v)) \\ &\quad - E(B^-(v) \mid \mathcal{F}(u, v)). \end{aligned} \quad (3.9)$$

The first term is a planar martingale since

$$\begin{aligned} E(\bar{M}(1) \mid \mathcal{F}(u, v)) &= E(E(\bar{M}(1) \mid \mathcal{F}(1, v)) \mid \mathcal{F}(u, v)) \\ &= E(\bar{M}(v) \mid \mathcal{F}(u, v)) \text{ a.e.} \end{aligned}$$

For fixed v , the other terms are martingales in u . For if $u \leq u'$, $E(E(B^+(v) \mid \mathcal{F}(u', v)) \mid \mathcal{F}(u, v)) = E(B^+(v) \mid \mathcal{F}(u, v))$. For fixed u , the other terms are submartingales in v . For if $v \leq v'$,

$$\begin{aligned} E(E(B^+(v') \mid \mathcal{F}(u, v')) \mid \mathcal{F}(u, v)) &\geq E(E(B^+(v) \mid \mathcal{F}(u, v')) \mid \mathcal{F}(u, v)) \\ &= E(B^+(v) \mid \mathcal{F}(u, v)) \text{ a.e.} \end{aligned}$$

The last two terms on the right hand side of (3.9) are thus V -martingales and A -submartingales; therefore, they are positive H -submartingales. They are L^1 right continuous by the right continuity of B^+ and B^- , Theorem 1.5, and the right continuity of the increasing family $\{\mathcal{F}(t)\}$.

Substituting into (3.8)

$$\begin{aligned} M(u, v) &= \{E(M(1, 1) \vee 0 \mid \mathcal{F}(u, v)) + E(\bar{M}(1) \vee 0 \mid \mathcal{F}(u, v)) + E(B^+(v) \mid \mathcal{F}(u, v))\} \\ &\quad - \{-E(M(1, 1) \wedge 0 \mid \mathcal{F}(u, v)) - E(\bar{M} \wedge 0 \mid \mathcal{F}(u, v)) + E(B^-(v) \mid \mathcal{F}(u, v))\} \\ &\quad + E(M(u, 1) \mid \mathcal{F}(u, v)). \end{aligned}$$

The first and second terms in the braces are positive, L^1 -right continuous H -submartingales. The last term can be decomposed as above, thus $M = M^+ - M^-$ where M^+ and M^- are positive L^1 -right continuous H -submartingales. Setting $X = (M^+ + A^+) - (M^- + A^-)$ gives the desired decomposition.

(2) implies (1) follows easily since Y and Z each belongs to class D' and class D on Γ_1 and Γ_2 .

The next corollary shows a distinguishing feature of class D' V -processes when compared to Theorem 2.4.

COROLLARY 3.7. *Let $\{X(t), \mathcal{F}(t), t \in T\}$ be an L^1 -right continuous V -process in class D' , then $X = M + Y - Z$ where M is L^1 -right continuous V -martingale and Y and Z are positive L^1 -right continuous H -submartingales.*

Proof. $X = M + \bar{X}$ where \bar{X} is the normal H -process associated with X . \bar{X} belongs to class D on Γ_1 and Γ_2 since it equals zero on Γ_u . By Theorem 3.6, $\bar{X} = Y - Z$.

4. INEQUALITIES AND CONVERGENCE

For this section a further condition will be placed on all increasing families $\{\mathcal{F}(t), t \in T\}$. Namely, if s and t are in T , the operators $E(\cdot | \mathcal{F}(t))$ shall satisfy

$$\begin{aligned} E(\cdot | \mathcal{F}(s \wedge t)) &= E(E(\cdot | \mathcal{F}(s)) | \mathcal{F}(t)) \\ &= E(E(\cdot | \mathcal{F}(t)) | \mathcal{F}(s)). \end{aligned} \quad (4.1)$$

(4.1) is equivalent to $\mathcal{F}(s)$ and $\mathcal{F}(t)$ being conditionally independent with respect to $\mathcal{F}(s \wedge t)$. (4.1) has been used extensively by Cairoli and Walsh [3]. They point out that two important cases where (4.1) is satisfied are (1) $\mathcal{F}(s^1, s^2) = \mathcal{H}(s^1) \otimes \mathcal{G}(s^2)$ is a product of two independent increasing families on $[0, 1]$, and (2) $\mathcal{F}(s) = \sigma\{X(t): t \leq s\}$ where X has independent increments.

The following definition is recalled from the general theory of processes:

DEFINITION 4.1. A function $X: T \times \Omega \rightarrow \mathbb{R}$ is *separable* if there exists a countable dense set $N \subseteq T$ and $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 0$ such that for an arbitrary open set $U \subseteq T$ and arbitrary closed set $G \subseteq \mathbb{R}$, the two sets

$$\{\omega: X(t, \omega) \in G, t \in U\}$$

and

$$\{\omega: X(t, \omega) \in G, t \in U \cap N\}$$

differ from each other by only a subset of Ω_0 .

Separability is not a stringent condition; in fact, by Theorem 2, p. 153 and Theorem 5, p. 155 of Gikhman and Skorokhod [9], if $\{X(t), t \in T\}$ is L^1 -right continuous, there always exists a separable modification $\{X'(t), t \in T\}$ and the set N , called the *set of separability*, can be any countable dense subset since T is a separable (locally compact) metric space.

The next inequality is due to Cairoli [2] for planar martingales.

THEOREM 4.2. Let $\{X(t), \mathcal{F}(t), t \in T\}$ be a separable process with $D = \{(i/2^n, j/2^n): 0 \leq i \leq 2^n, 0 \leq j \leq 2^n, n \geq 0\}$ as its set of separability. Setting $\Phi(x) = |x| \log^+ x$, let X satisfy

$$\Phi_v = \sup \left\{ E \left(\Phi \cdot \sum_{t \in g} |E(\Delta_g X(t) | \mathcal{F}(t))| \right) : g \text{ a grid on } T \right\} < \infty \quad (4.2)$$

and

$$\begin{aligned} \Phi_j = \sup \left\{ E \left(\Phi \cdot \sum_{i=0}^{n-1} |E(X(t_{i+1}) - X(t_i) | \mathcal{F}(t_i))| \right) : t_i < t_{i+1}, t_i \in \Gamma_j \right\} < \infty \\ \text{for } j = 1 \text{ and } 2. \end{aligned} \quad (4.3)$$

Then, there exist real constants $a > 0$ and $b > 0$ depending only on Φ_V , Φ_1 and Φ_2 such that

$$P\{\sup_{t \in T} |X_t| > \lambda\} \leq \frac{1}{\lambda} (a + bE(\Phi \cdot X(1, 1))). \quad (4.4)$$

Proof. For $x \geq 0$ and $y \geq 0$, Φ satisfies

$$x \leq \Phi(x) + e \quad (4.5)$$

$$\Phi(x + y) \leq 4(\Phi(x) + \Phi(y)) + \log 16. \quad (4.6)$$

These inequalities are crude, but will suffice since finding the smallest possible a and b for (4.4) will not be attempted here. (4.2), (4.3) and (4.5) imply immediately that X is an H -process.

Fix $n \geq 0$. Set $A_{ij} = 0$ if $i = 0$ or $j = 0$ and

$$A_{ij} = \sum_{k=0}^{i-1} \sum_{l=0}^{j-1} E\left(\Delta X\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \middle| \mathcal{F}\left(\frac{k}{2^n}, \frac{l}{2^n}\right)\right) \\ \text{for } 1 \leq i \leq 2^n, \quad 1 \leq j \leq 2^n$$

Set $M_{ij} = X(i/2^n, j/2^n) - A_{ij}$. This is the discrete Doob decomposition of $\{X(i/2^n, j/2^n)\}$. M satisfies $E(\Delta M_{ij} | \mathcal{F}(i/2^n, j/2^n)) = 0$ a.c. where $\Delta M_{ij} = M_{i+1, j+1} - M_{i, j+1} - M_{i+1, j} + M_{ij}$, and A satisfies $\sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} E(|\Delta A_{kl}|) \leq \Phi_V + e$. By the above property,

$$M_{ij} = E\left(M_{2^n, j} \middle| \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right) + E\left(M_{i, 2^n} \middle| \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right) \\ - E\left(M_{2^n, 2^n} \middle| \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right) \quad \text{a.e.} \quad (4.7) \\ = M_{ij}^1 + M_{ij}^2 + M_{ij}^3.$$

$\{M_{ij}^1, \mathcal{F}(i/2^n, j/2^n), 0 \leq i \leq 2^n\}$ is a discrete-time martingale for each $0 \leq j \leq 2^n$. In fact, $\{M_{ij}^1, \mathcal{F}(i/2^n, 1), 0 \leq i \leq 2^n\}$ is a martingale for each j by (4.1), since

$$E\left(M_{i+1, j}^1 \middle| \mathcal{F}\left(\frac{i}{2^n}, 1\right)\right) = E\left(E\left(M_{i+1, j}^1 \middle| \mathcal{F}\left(1, \frac{j}{2^n}\right)\right) \middle| \mathcal{F}\left(\frac{i}{2^n}, 1\right)\right) \quad \text{a.e.} \\ = E\left(M_{i+1, j}^1 \middle| \mathcal{F}\left(\frac{i}{2^n}, \frac{j}{2^n}\right)\right) = M_{ij}^1 \quad \text{a.e.}$$

Set $S_i = \sup_j |M_{ij}^1|$; $\{S_i, \mathcal{F}(i/2^n, 1), 0 \leq i \leq 2^n\}$ is a positive submartingale being the maximum of a finite number of positive submartingales over $\{\mathcal{F}(i/2^n, 1)\}$.

$$P\{\sup_{i, j} |M_{i, j}^1| > \lambda\} = P\{\sup_i S_i > \lambda\} \leq \frac{1}{\lambda} E(S_{2^n}). \quad (4.8)$$

The last inequality is Doob's maximal inequality (cf. Doob (1953)). Consider $\{M_{2^n, j}^1, \mathcal{F}(1, j/2^n), 0 \leq j \leq 2^n\}$. By (4.7)

$$\begin{aligned} & E\left(\sum_{j=0}^{2^n-1} \left| E\left(M_{2^n, j+1}^1 - M_{2^n, j}^1 \mid \mathcal{F}\left(1, \frac{j}{2^n}\right)\right) \right|\right) \\ &= E\left(\sum_{j=0}^{2^n-1} \left| E\left(M_{2^n, j+1} - M_{2^n, j} \mid \mathcal{F}\left(1, \frac{j}{2^n}\right)\right) \right|\right) \\ &\leq E\left(\sum_{j=0}^{2^n-1} \left| E\left(X\left(1, \frac{j+1}{2^n}\right) - X\left(1, \frac{j}{2^n}\right) \mid \mathcal{F}\left(1, \frac{j}{2^n}\right)\right) \right|\right) \\ &\quad + E\left(\sum_{j=0}^{2^n-1} |A_{2^n, j+1} - A_{2^n, j}|\right) \\ &\leq \Phi_2 + \Phi_V + 2e. \end{aligned}$$

Set $B_0 = 0$, $B_j = \sum_{k=0}^{j-1} E(M_{2^n, k+1}^1 - M_{2^n, k}^1 \mid \mathcal{F}(1, k/2^n))$, and $N_j = M_{2^n, j}^1 - B_j$. $\{N_j, \mathcal{F}(1, j/2^n), 0 \leq j \leq 2^n\}$ is a martingale.

$$\begin{aligned} E(S_{2^n}) &= E(\sup_j |M_{2^n, j}^1|) \leq E(\sup_j |N_j|) + E(\sup_j |B_j|) \\ &\leq \frac{e}{e-1} + \frac{e}{e-1} E(\Phi \cdot N_{2^n}) + \Phi_2 + \Phi_V + 2e. \quad (4.9) \\ &\leq a_1 + b_1 E(4(\Phi \cdot M_{2^n, 2^n}^1 + \Phi \cdot B_n) + \log 16) \\ &\leq a_1 + b_1 (E(4\Phi \cdot M_{2^n, 2^n} + 16\Phi_V + 16\Phi_2) + 5 \log 16) \\ &= a_2 + b_2 E(\Phi \cdot M_{2^n, 2^n}), \quad (4.10) \end{aligned}$$

where a_2 and b_2 depend only on Φ_V and Φ_2 . The first two terms of (4.9) are from a well known inequality of Doob. Substituting (4.10) into (4.8) gives

$$P(\sup_{ij} |M_{ij}^1| > \lambda) \leq \frac{1}{\lambda} (a_1 + b_2 E(\Phi \cdot M_{2^n, 2^n})). \quad (4.11)$$

The same inequality must hold for M^2 except for possibly larger constants which depend also on Φ_1 . Also (4.11) holds for M^3 with $a_1 = b_2 = e/(e+1)$. This is Cairoli's [2] result for planar martingales.

Since $X(i/2^n, j/2^n) = M_{ij} + A_{ij}$, combining (4.7) and (4.11) and adjusting constants gives

$$P\left(\sup_{t \in (i/2^n, j/2^n)} |X(t)| > \lambda\right) \leq \frac{1}{\lambda} (a_2 + b_3 E(\Phi \cdot M_{2^n, 2^n})) \quad (4.12)$$

$$\begin{aligned} E(\Phi \cdot M_{2^n, 2^n}) &\leq 4E(\Phi \cdot X(1, 1)) + 4E(\Phi \cdot A_{2^n, 2^n}) + \log 16 \\ &\leq 4E(\Phi \cdot X(1, 1)) + 4\Phi_V + 4e + \log 16. \quad (4.13) \end{aligned}$$

Substituting (4.13) into (4.12) and adjusting constants gives

$$P\left(\sup_{t \in (i/2^n, j/2^n)} |X(t)| > \lambda\right) \leq \frac{1}{\lambda} (a + bE(\Phi \cdot X(1, 1))). \quad (4.14)$$

The right side of (4.14) is independent of n . Letting $n \rightarrow \infty$ shows (4.14) holds for the sup taken over $t \in D$. (4.4) then follows by separability and the proof is complete.

The next result is an improvement of Theorem 1.5 for a class of H -processes in LlogL.

THEOREM 4.3. *Let $\{X(t), \mathcal{F}(t), t \in T\}$ be a separable, L^1 -right continuous H -process. Setting $\Phi(x) = |x| \log^+ x$; let X be such that $E(\Phi \cdot X(1, 1))$, Φ_ν , Φ_1 and Φ_2 are all finite for Φ_ν , Φ_1 and Φ_2 defined by (4.2) and (4.3). Then for each $t \in T - \Gamma_1$,*

$$\lim_{s \rightarrow t, s \ll t} X(s) \text{ exists a.e. and in } L^1\text{-mean.}$$

Proof. There is no loss of generality in restricting the proof to $t = (1, 1)$. By Theorem 3, $X = M + A$ where M is a V -martingale and almost all maps $t \mapsto A(t, \omega)$ are of Hardy bounded variation. It is clear that $\lim_{s \rightarrow (1, 1), s \ll (1, 1)} A(s)$ exists a.e. and in L^1 .

Now,

$$\begin{aligned} M(s) &= M(u, v) = E(M(1, v) | \mathcal{F}(u, v)) + E(M(u, 1) | \mathcal{F}(u, v)) \\ &\quad - E(M(1, 1) | \mathcal{F}(u, v)) \\ &= E(M(1, v) | \mathcal{F}(u, 1)) + E(M(u, 1) | \mathcal{F}(1, v)) - E(M(1, 1) | \mathcal{F}(u, v)), \\ &\quad \text{by (4.1),} \\ &= M^1(s) + M^2(s) + M^3(s) \quad (\text{say}). \end{aligned}$$

To analyze the limit behavior of $M^i(\cdot)$, $i = 1, 2$ or 3 , the following lemma will be useful.

LEMMA. *Let $\{\mathcal{F}_t, t \in [0, 1]\} \subseteq \mathcal{F}$ be an increasing family of σ -subalgebras and $\mathcal{F}_{1-} = \sigma(\bigcup \mathcal{F}_t)$. Let $\{X_t, t \in [0, 1]\}$ be a set of \mathcal{F} -measurable separable random variables such that $|X_t| \leq \alpha$ a.e. ($\alpha < \infty$) and $\lim_{t \rightarrow 1} X_t = X$ a.e., then $\lim_{s \rightarrow 1, t \rightarrow 1} E(X_t | \mathcal{F}_s) = E(X | \mathcal{F}_{1-})$ a.e.*

Proof. Since $\lim_{t \rightarrow 1} E(X_t | \mathcal{F}_{1-}) = E(X | \mathcal{F}_{1-})$ a.e. by the dominated convergence theorem for conditional expectations, without loss of generality X may be assumed \mathcal{F}_{1-} -measurable. Set $Y_{t_0} = \sup_{t_0 \leq t < 1} |X_t - X| \leq 2\alpha$.

$$\begin{aligned} \overline{\lim}_{\substack{s \rightarrow 1 \\ t \rightarrow 1}} |X - E(X_t | \mathcal{F}_s)| &\leq \lim_{s \rightarrow 1} |X - E(X | \mathcal{F}_s)| + \overline{\lim}_{\substack{s \rightarrow 1 \\ t \rightarrow 1}} E(|X_t - X| | \mathcal{F}_s) \\ &\leq \lim_{s \rightarrow 1} E(Y_{t_0} | \mathcal{F}_s) = E(Y_{t_0} | \mathcal{F}_{1-}) \quad \text{a.e.,} \end{aligned}$$

by the martingale convergence theorem. Letting $t_0 \rightarrow 1$, $E(Y_{t_0} | \mathcal{F}_{1-}) \rightarrow 0$ a.e. by the dominated convergence theorem for conditional expectations. This completes the proof of the lemma.

$\{M(1, v), \mathcal{F}(1, v), 0 \leq v \leq 1\}$ is a semi-martingale. By a theorem of K. M. Rao [11], $M(1, v) = S(v) - S'(v)$ where S and S' are positive supermartingales. (This theorem is the linear index analog of Theorem 2.4.) Set $N(u, v) = E(S(v) | \mathcal{F}(u, 1))$ and $N^\lambda(u, v) = E(S(v) \wedge \lambda | \mathcal{F}(u, 1))$. N and N^λ are both H -processes which satisfy the hypotheses of Theorem 4.2. N^λ is an H -process since $S \wedge \lambda$ is a supermartingale. By the lemma, $\lim_{(u,v) \rightarrow (1,1)} N^\lambda(u, v)$ exists a.e. By Theorem 4.2, for $\delta > 0$,

$$P\{\sup_{(u,v)} |N^\lambda(u, v) - N(u, v)| > \epsilon\} \leq \frac{1}{\delta\epsilon} (a + bE(\Phi \cdot (\delta(S(1) - S(1) \wedge \lambda)))). \quad (4.15)$$

For any $\epsilon > 0$, the right side of (4.15) can be made arbitrarily small by taking a sufficiently large δ and then letting $\lambda \rightarrow \infty$. This shows $\lim_{(u,v) \rightarrow (1,1)} N(u, v)$ exists a.e. Applying the same argument to $\{S'(v)\}$ shows that $\lim_{s \rightarrow (1,1)} M'(s) = \lim_{(u,v) \rightarrow (1,1)} E(M(1, v) | \mathcal{F}(u, 1))$ exists a.e. This limit exists in L^1 -mean since $\{M(1, v); 0 \leq v < 1\}$ is uniformly integrable. To see this, note

$$\begin{aligned} |M(1, v)| &\leq |E(M(1, v) - M(1, 1) | \mathcal{F}(1, v))| + |E(M(1, 1) | \mathcal{F}(1, v))| \\ E(\Phi \cdot M(1, v)) &\leq 4\Phi_2 + 4E(\Phi \cdot M(1, 1)) + \log 16 \\ &\leq 4\Phi_2 + 4(4E(\Phi \cdot X(1, 1)) + 4\Phi_\nu + \log 16) + \log 16. \end{aligned}$$

The proof of the existence of a limit for M^2 is the same, and the proof for M^3 is similar, but easier. This completes the proof of the theorem.

Cairolì [2] proved Theorem 4.3 when X is a planar martingale. He also constructed a uniformly integrable martingale M satisfying $E(|M(1, 1)|) < \infty$ and $E(\Phi \cdot M(1, 1)) = \infty$ for which $\lim_{(u,v) \rightarrow (1,1)} M(u, v)$ does not exist.

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